Completely Isometric Maps and Triangular Operator Algebras

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INTRODUCTION

This paper is divided into three sections. The first section consists of a survey of some known results in the theory of non-self-adjoint operator algebras. Almost all the definitions and results required for the remainder of the paper are stated here; in addition a "spatial implementation" type of theorem proven by J. Ringrose is stated in order to provide a proper perspective for section II.

The main result of section II is a theorem which asserts that an identity preserving completely isometric linear map between certain non-self-adjoint algebras of operators, e.g. nest algebras and non-irreducible maximal triangular algebras, must be implemented by a unitary transformation of the underlying Hilbert spaces. The theorem is obtained by proving that the identity representation of the C*-algebra generated by such a non-self-adjoint algebra is a boundary representation and then applying general results about boundary representations by Arveson [1].

The third section consists of a study of the ideal theory for a special class of maximal triangular algebras, the ordered bases. This section investigates the relationship between the radical of the algebra and two special types of ideals, each partially analogous to the ideal of strictly upper triangular matrices in M_n .

Throughout this paper we use the following notation and conventions. Hilbert space is always assumed to be complex and separable. With $\mathcal{H}$ a Hilbert space, $\mathcal{B}(\mathcal{H})$ is the set of all bounded linear operators on $\mathcal{H}$. The ordering of projections in $\mathcal{B}(\mathcal{H})$ is the usual one, viz. $E \leq F$ if and only if $EF = E$. All ideals are two sided and ideals in C*-algebras are also assumed to be closed (in the norm topology). The usual bracket notation is used for closed linear spans. For any set $\mathcal{T}$ of operators, $C^*(\mathcal{T})$ will denote the C*-algebra generated by $\mathcal{T}$.

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I. PRELIMINARIES

Triangular Operator Algebras. In [5] Kadison and Singer introduced a class of non self-adjoint operator algebras on Hilbert space. These algebras generalize to infinite dimensional space the notion of "algebras of upper triangular matrices." The following is a summary of some of the ideas and results in the paper by Kadison and Singer.

Definition 1. If $\mathfrak{A}$ is a maximal abelian self-adjoint subalgebra of $\mathfrak{B}(\mathfrak{H})$, a subalgebra, $\mathfrak{T}$, of $\mathfrak{B}(\mathfrak{H})$ is said to be *triangular* with *diagonal* $\mathfrak{A}$ provided $\mathfrak{T} \cap \mathfrak{T}^* = \mathfrak{A}$. If $\mathfrak{T}$ is not a proper subalgebra of another triangular algebra then $\mathfrak{T}$ is said to be *maximal triangular*. A projection in $\mathfrak{B}(\mathfrak{H})$ which is invariant under $\mathfrak{T}$ is called a *hull* of $\mathfrak{T}$.

It is easily seen that each hull of a triangular algebra lies in the diagonal of that algebra and that if one triangular algebra contains another then both have the same diagonal. Furthermore, a Zorn's Lemma argument can be used to show that if $\mathfrak{T}$ is a triangular algebra then $\mathfrak{T}$ is contained in some maximal triangular algebra.

It might be supposed, from the finite dimensional case, that the hulls of a maximal triangular algebra are totally ordered and that these hulls generate the diagonal (as a von-Neumann algebra). The second supposition is false: Kadison and Singer provide an example of an irreducible triangular algebra, i.e., one whose only hulls are the projections 0 and I. The first one is true and is proven with the aid of the following useful lemma:

Lemma 2. Let $\mathfrak{T}$ be a maximal triangular algebra with diagonal $\mathfrak{D}$. Let E be a hull of $\mathfrak{T}$ and F a projection in $\mathfrak{D}$ orthogonal to E. If T is an operator in $\mathfrak{B}(\mathcal{H})$ such that $T = ETF$ then T lies in $\mathfrak{T}$.

The most incisive results about triangular algebras can be obtained for those with "sufficiently many" hulls.

Definition 3. A triangular algebra whose hulls generate the diagonal is said to be hyperreducible. A maximal hyperreducible triangular algebra is called an ordered basis.

Examples of ordered bases may be obtained as follows: Let ρ be a Borel probability measure on $[0,1]$ and let $\mathcal{H}$

be the Hilbert space $L^2([0,1],\rho)$. For each function f in $L^\infty([0,1],\rho)$ let L_f be the bounded linear operator acting on $\mathcal{H}$ by "multiplication by f." The set of all such operators, $\mathcal{D}$, is a maximal abelian self-adjoint subalgebra of $\mathfrak{B}(\mathcal{H})$. For each $\lambda \in [0,1]$ denote by E_λ and E_{λ^-} the operators in $\mathcal{D}$ corresponding to the characteristic functions of $[0,\lambda]$ and $[0,\lambda)$ respectively. $\{E_\lambda, E_{\lambda^-}\}_{\lambda\in[0,1]}$ is a totally ordered family of projections in $\mathcal{D}$ which generates $\mathcal{D}$. Let $\mathfrak{J}$ be the set of all operators in $\mathfrak{B}(\mathcal{H})$ which leave invariant each projection E_λ and E_{λ^-}. Then $\mathfrak{J}$ is an ordered basis with diagonal $\mathcal{D}$ and hulls $\{E_{\lambda^-}, E_\lambda\}_{\lambda\in[0,1]}$.

Definition 4. An ordered basis of this form shall be referred to as a standard ordered basis.

Ordered bases are studied in considerable detail by Kadison and Singer. Two of their most important results are the following:

Theorem 5. Let $\{E_\alpha\}$ be a totally ordered family of projections in $\mathfrak{B}(\mathcal{H})$ which generates a maximal abelian self-adjoint subalgebra, $\mathcal{D}$, of $\mathfrak{B}(\mathcal{H})$. Then the set $\mathfrak{J}$ of all operators in $\mathfrak{B}(\mathcal{H})$ which leave invariant each E_α is an ordered basis with diagonal $\mathcal{D}$. If $\{E_\alpha\}$ is closed under unions and intersections then $\{E_\alpha\}$ is the set of hulls of $\mathcal{J}$. Each ordered basis arises in this way.

Theorem 6. (Representation theorem.) Each ordered basis is unitarily equivalent to a standard ordered basis.

Nest Algebras. John Ringrose [9,10] has studied a class of non self-adjoint algebras closely related to triangular algebras.

Definition 7. A family $\mathcal{P}$ of projections in $\mathcal{B}(\mathcal{H})$ will be called a nest if $\mathcal{P}$ is totally ordered. A nest $\mathcal{P}$ is complete provided

(i) $0, I \in \mathcal{P}$

(ii) $\mathcal{P}$ is closed under unions and intersections.

A nest is maximal if it is not a proper subnest of another nest.

If $\mathcal{P}$ is a complete nest and $E \in \mathcal{P}$, define $E_- = \vee \{F \in \mathcal{P} \mid F < E\}$. If E has an immediate predecessor then E_- is that predecessor; otherwise $E_- = E$.

A nest is maximal if and only if the dimension of the range of $E-E_-$ is at most one, for each E in the nest.

(See [9], Lemma 2.1.)

Let $\mathcal{P}$ be a nest. A Zorn's lemma argument provides the existence of a maximal (and, in particular, a complete) nest containing $\mathcal{P}$. Since the intersection of complete nests is complete, we can deduce the existence of a smallest complete nest containing $\mathcal{P}$, which will be denoted by $co(\mathcal{P})$.

<u>Definition 8</u>. Let $\mathcal{P}$ be a nest. The <u>nest algebra</u> associated with $\mathcal{P}$ is the algebra, $\mathfrak{A}_{\mathcal{P}}$, of all operators in $\mathfrak{B}(\mathcal{H})$ which leave invariant each projection of $\mathcal{P}$.

It is easy to see that $\mathfrak{A}_{\mathcal{P}} = \mathfrak{A}_{co(\mathcal{P})}$, and for this reason we shall henceforth assume that all nests are complete. When no confusion can result, the subscript $\mathcal{P}$ will be dropped. Note that the intersection of the class of nest algebras with the class of triangular algebras is precisely the class of ordered bases. (See Theorem 5.)

It will be necessary later to determine whether or not a given operator in a nest algebra lies in the radical of the algebra. One of Ringrose's theorems provides an effective criterion.

<u>Theorem 9</u>. Let $\mathcal{P}$ be a complete nest in $\mathfrak{B}(\mathcal{H})$, let $\mathfrak{n}$ be the nest algebra associated with $\mathcal{P}$ and let $\mathfrak{R}$ be the radical of $\mathfrak{n}$. Let $T \in \mathfrak{n}$. Then, in order that $T \in \mathfrak{R}$, it is necessary and sufficient that the following condition be satisfied: given any real number $\varepsilon > 0$ there exists a finite subnest $(E_o, E_1, \ldots, E_n)$ of $\mathcal{P}$ such that
$0 = E_o < E_1 < \cdots < E_n = I$ and
$\|(E_j - E_{j-1})T(E_j - E_{j-1})\| < \varepsilon \qquad (j = 1,2,\ldots,n).$

The condition in this theorem will be referred to in this paper as "Ringrose's Criterion."

In his papers, Ringrose makes very effective use of the rank one operators which lie in a given nest algebra or triangular algebra. Let us first introduce some notation: if x and y are vectors in $\mathcal{H}$ let $x \otimes y$ denote the operator in $\mathfrak{B}(\mathcal{H})$ defined by $x \otimes y(w) = (w,x)y$ for all $w \in \mathcal{H}$. $x \otimes y$ is a rank one operator whose norm is $\|x\| \cdot \|y\|$. Any rank one operator may be put in this form, and the one dimensional subspaces generated by y and x are uniquely determined (as the range and the orthogonal complement of the kernel, respectively). The following lemma makes it easy to determine whether the rank one operator $x \otimes y$ lies in a given nest algebra.

<u>Lemma 10</u>. Let $\mathfrak{p}$ be a complete nest and let $\mathfrak{n}$ be the associated nest algebra. Let x and y be non-zero vectors in $\mathcal{H}$. Then $x \otimes y \in \mathfrak{n}$ if and only if there is a projection E in $\mathfrak{p}$ such that $Ey = y$ and $E_- x = 0$.

The rank one operators in a nest algebra are sufficient to determine the complete nest of invariant projections associated with the nest algebra.

<u>Lemma 11</u>. Let $\mathfrak{n}$ be a nest algebra associated with the complete nest $\mathfrak{p}$ and let E be a projection which is invariant under each rank one operator in $\mathfrak{n}$. Then E lies in $\mathfrak{p}$.

The following two definitions are found in [10].

<u>Definition 12</u>. Let $\mathfrak{n}$ be a nest algebra and let $\mathfrak{m}$ be a subalgebra of $\mathfrak{n}$. We say that $\mathfrak{m}$ is a <u>large subalgebra</u> of $\mathfrak{n}$ if

(i) $\mathfrak{m}$ contains each rank one operator in $\mathfrak{n}$

(ii) $\mathfrak{m}$ contains at least one maximal abelian self-adjoint subalgebra of $\mathfrak{B}(\mathcal{H})$.

<u>Definition 13</u>. A maximal triangular algebra whose hull nest is a maximal nest is said to be <u>strongly reducible</u>.

Note that an ordered basis is strongly reducible. Also, a strongly reducible triangular algebra $\mathfrak{J}$ with hull nest $\mathcal{P}$ is a large subalgebra of the nest algebra $\mathfrak{n}_{\mathcal{P}}$.

A theorem of Ringrose related to the main result in this paper concerns algebraic isomorphisms of strongly reducible triangular algebras.

Definition 14. Let $\mathfrak{J}$ and $\mathcal{S}$ be any algebras of operators on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ respectively. An algebraic isomorphism from $\mathfrak{J}$ on to $\mathcal{S}$ is a bijective linear and multiplicative mapping φ from $\mathfrak{J}$ onto $\mathcal{S}$. If P is a bijective bicontinuous linear mapping from $\mathcal{K}$ onto $\mathcal{H}$ such that $\varphi(T) = P^{-1}TP$ $(T \in \mathfrak{J})$ then we say φ is spatial and is implemented by P. If P is unitary, i.e., is an isometry, then we say φ is unitarily implemented.

Neither any continuity properties nor any relation to the adjoint operation is assumed for algebraic isomorphisms. The theorem mentioned above is the following ([10], Theorems 5.5 and 4.1):

Theorem 15. Let $\mathfrak{J}_1$, and $\mathfrak{J}_2$ be maximal triangular algebras, with diagonals $\mathcal{D}_1$ and $\mathcal{D}_2$, acting on Hilbert spaces $\mathcal{H}_1$, and $\mathcal{H}_2$. Let $\varphi\colon \mathcal{J}_1 \longrightarrow \mathcal{J}_2$ be an algebraic isomorphism.

(1) If $\mathcal{J}_1$ and $\mathcal{J}_2$ are ordered bases then φ is spatial.

(2) If $\mathcal{J}_1$ and $\mathcal{J}_2$ are strongly reducible and $\varphi(\mathcal{D}_1) = \mathcal{D}_2$ then φ is spatial. Furthermore, φ is implemented by an operator of the form $P = DU$ where U is a unitary operator from $\mathcal{H}_2$ onto $\mathcal{H}_1$ and $D \in \mathcal{D}_1$.

<u>Boundary Representations</u>. In [1] Arveson has studied the question: to what extent does an algebra of operators on a Hilbert space determine the structure of the C*-algebra it generates? To answer this question he uses a non-commutative generalization of Choquet boundary and Silov boundary. Before defining these generalizations and describing their use, we must first introduce some terminology and state a theorem of Stinespring.

If $\mathcal{J}$ is a vector space of operators on a Hilbert space $\mathcal{H}$ and M_n is the algebra of $n \times n$ complex matrices, then $\mathcal{J} \otimes M_n$, the set of $n \times n$ matrices with entries in $\mathcal{J}$, is a linear subspace of the space operators on the Hilbert space $\mathcal{H} \oplus \mathcal{H} \oplus \ldots \oplus \mathcal{H}$ (n factors). If

$\varphi: \mathfrak{J} \longrightarrow \mathfrak{S}$ is a linear map from one vector space of operators into another, then for each positive integer n, define $\varphi_n: \mathfrak{J} \otimes M_n \longrightarrow \mathfrak{S} \otimes M_n$ by applying φ element by element to each matrix over $\mathfrak{J}$, i.e., $\varphi_n(T_{ij}) = (\varphi(T_{ij}))$.

Definition 16. Let $\varphi: \mathfrak{J} \longrightarrow \mathfrak{S}$ be linear. We say φ is completely positive (resp. completely isometric) provided each φ_n is positive (resp. isometric).

(If $\mathfrak{J}$ and $\mathfrak{S}$ are linear subspaces of $\mathfrak{B}(\mathcal{H})$ and $\mathfrak{B}(\mathcal{K})$ respectively then $\varphi: \mathfrak{J} \longrightarrow \mathfrak{S}$ is positive if $\varphi(T)$ is a positive operator whenever T is a positive operator. The possibility that $\mathfrak{J}$ has no positive operators in it is not excluded. φ is isometric provided $\|\varphi(T)\| = \|T\|$ for all $T \in \mathfrak{J}$.)

If B is an arbitrary C*-algebra then $B \otimes M_n$ is a *-algebra (define * by $(X_{ij})^* = (X_{ji}{}^*))$ which possesses a unique C*-norm. (See, for example, [1] p. 143.) We may therefore define completely positive and completely isometric linear maps on B or on linear subspaces of B.

Examples of completely positive linear maps on C*-algebras may be obtained as follows: let B be a C*-algebra with identity, let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, let π be a representation of B on $\mathcal{K}$ and let V be a bounded linear operator from $\mathcal{H}$ to $\mathcal{K}$. Define $\varphi: B \longrightarrow \mathfrak{B}(\mathcal{H})$ by $\varphi(X) = V^*\pi(X)V$ for all $X \in B$. Then φ is a completely positive linear mapping of B into $\mathfrak{B}(\mathcal{H})$. Stinespring's theorem asserts that, for C*-algebras with identity, every completely positive linear mapping into $\mathfrak{B}(\mathcal{H})$ is of this form. Explicitly,

Theorem 17. Let B be a C*-algebra with identity and let $\mathcal{H}$ be a Hilbert space. Then every completely positive linear map of B into $\mathfrak{B}(\mathcal{H})$ has the form $\varphi(X) = V^*\pi(X)V$, where π is a representation of B on some Hilbert space $\mathcal{K}$ and V is a bounded linear operator from $\mathcal{H}$ to $\mathcal{K}$. We may further assume that $[\pi(B)V\mathcal{H}] = \mathcal{K}$.

Let B be a C*-algebra with identity and let $\mathbf{A}$ be a linear subspace of B which generates B as a C*-algebra, i.e., $B = C^*(A)$.

Definition 18. An irreducible representation π of B is called a boundary representation for A if $\pi|A$ has a unique completely positive linear extension to B.

Note that π itself is always a completely positive extension of $\pi|A$, so π is a boundary representation if it is the only linear extension which is completely positive. If $B = C(X)$ where X is a compact Hausdorff space and A is a separating linear subspace of $C(X)$ then irreducible representations of B correspond to points of X and boundary representations correspond to points in the Choquet boundary of X relative to A.

Definition 19. Let B and A be as above and assume further that the identity of B lies in A. A closed two-sided ideal J in B is called a boundary ideal for A if the canonical quotient map $q: B \longrightarrow B/J$ is completely isometric on A. A boundary ideal is called the Silov boundary for A if it contains every other boundary ideal.

In the commutative case, i.e., when B is a $C(X)$, closed ideals in B correspond to closed subsets of X, boundary ideals correspond to subsets which are a boundary for A and the Silov boundary ideal corresponds to the usual Silov boundary for A. (See [1] for details.) The Silov boundary of Definition 19 is unique when it exists, but it is not yet known whether it always does exist. Arveson does show that the Silov boundary exists for "admissible" subspaces, as defined below. B and A satisfy the same hypotheses as in Definition 19.

Definition 20. A is called an *admissible subspace* of B if the intersection of the kernels of the boundary representations (for A) is a boundary ideal for A.

Arveson proves that any boundary ideal for A is contained in the kernel of any boundary representation for A (where A need not be admissible). It follows that when A is admissible the intersection of all kernels of boundary representations is the Silov boundary ideal for A.

Note that if B is an irreducible C*-subalgebra of $\mathfrak{B}(\mathfrak{H})$, if A is a generating subspace of B such that $I \in A$, and if the identity representation of B is a boundary representation for A then A is admissible and $\{0\}$ is the Silov boundary for A. This is the context in which we shall apply the following theorem of Arveson.

Theorem 21. Let A_1 and A_2 be admissible subspaces of C*-algebras B_1 and B_2 respectively, with identities denoted by e_1 and e_2. Assume that both A_1 and A_2 have trivial Silov boundary ideals. Then every completely isometric linear map of A_1 on A_2 which takes e_1 to e_2

is implemented by a *-isomorphism of B_1 on B_2.

The following lemma can undoubtedly be found in many places (see [2] for example). However, we give [1, Lemma 3.4.4.] as a reference, for it is stated there in the form in which we need it.

Recall [2, Prop. 2.10.4] that a non-degenerate representation of an ideal K in a C*-algebra B with identity has a unique extension to a representation of B.

Lemma 22. Let B be a C*-algebra with identity and let K be a closed two sided ideal in B. Then any representation π of B such that $\pi(K) \neq 0$ may be written as a direct sum $\pi = \pi_o \oplus \pi_1$ where π_o is a representation of B such that $\pi_o(K) = 0$ and π_1 is the unique extension to B of a non-degenerate representation of K.

Note that we may have $\pi_o = 0$ in this decomposition.

II. COMPLETELY ISOMETRIC MAPS

The purpose of this section is to prove that completely isometric linear maps between certain non-self-adjoint operator algebras are implemented by unitary transformations of the underlying Hilbert spaces. The main tool will be theorem 21. In order to be able to apply theorem 21 we first provide sufficient conditions on a linear subspace $\mathfrak{J}$ of operators that the identity representation of the C*-algebra generated by $\mathfrak{J}$ be a boundary representation for $\mathfrak{J}$.

Theorem 23. Let $\mathfrak{J}$ be a subspace of $\mathfrak{B}(\mathcal{H})$ and let $\mathcal{S}$ be the norm closure of $\mathfrak{J} + \mathfrak{J}^*$. Assume that:

(i) $I \in \mathcal{S}$

(ii) There is a non-trivial projection E such that if $x \in E(\mathcal{H})$ and $y \in (I - E)(\mathcal{H})$ then $y \otimes x \in \mathcal{S}$.

Then, the identity representation of $C^*(\mathfrak{J})$ is a boundary representation for $\mathfrak{J}$.

Proof. We must first show that $C^*(\mathfrak{J})$ is an irreducible C*-algebra. Let $P \neq 0, I$ be a projection in $\mathfrak{B}(\mathcal{H})$ and assume that P commutes with $C^*(\mathfrak{J})$. Note first that $PE \neq 0$. For, if $PE = 0$ then $0 \neq P \leq I - E$. If $x \in P(\mathcal{H})$ and $y \in E(\mathcal{H})$ then $x \otimes y \in C^*(\mathfrak{J})$ and $x \otimes y$

does not commute with P. Likewise $(I - E)(I - P) \neq 0$. Choose x_o such that $PEx_o \neq 0$; choose y_o such that $(I - E)(I - P)y_o \neq 0$. Let $x = Ex_o$ and $y = (I - E)(I - P)y_o$. Then $y \otimes x \in C^*(\mathfrak{J})$ and

$$y \otimes x((I - P)y_o) = ((I - P)y_o, y)x = ((I - P)y_o, (I - E)y)x$$
$$= (y,y)x = \|y\|^2 x \ .$$

Since $\|y\| \neq 0$ and $Px \neq 0$ we see that $y \otimes x$ does not leave $I - P$ invariant. But this contradicts the assumption that P commutes with $C^*(\mathfrak{J})$. So $C^*(\mathfrak{J})$ is irreducible. Thus the identity representation, Id, is an irreducible representation of $C^*(\mathfrak{J})$. We must show that $\mathrm{Id}|\mathfrak{J}$ has a unique completely positive linear extension to $C^*(\mathfrak{J})$. Let φ be any completely positive extension of $\mathrm{Id}|\mathfrak{J}$ to $C^*(\mathfrak{J})$. Note that since positive linear maps on C^*-algebras are continuous and adjoint preserving, φ agrees with the identity map on $\mathcal{S}$. By Stinespring's theorem [Theorem 17] there is a representation π of $C^*(\mathfrak{J})$ on a Hilbert space $\mathcal{K}$ and a bounded linear operator $V: \mathcal{H} \longrightarrow \mathcal{K}$ such that $\varphi(S) = V^*\pi(S)V$ for all $S \in C^*(\mathfrak{J})$ and $[\pi(C^*(\mathfrak{J}))V\mathcal{H}] = \mathcal{K}$. In particular, (#) $T = V^*\pi(T)V$ for all $T \in \mathcal{S}$.

To prove the theorem it will suffice to show that V is a unitary operator. For then, $\varphi = V^*\pi V$ is a representation of $C^*(\mathfrak{J})$ which agrees with Id on $\mathfrak{J}$, and hence everywhere. Since $I \in \mathcal{S}$ and $\pi(I) = I$, equation (#) yields $I = V^*V$. Therefore V is an isometry. Let P be orthogonal projection (in $\mathfrak{B}(\mathcal{H})$) on the range of V. ($P = VV^*$.) We need only prove that range $(V) = \mathcal{H}$. The proof of this will consist mainly of a series of observations.

<u>Observation 1</u>. If x and y are vectors in $\mathcal{H}$ such that $T = x \otimes y \in \mathcal{S}$ then $P\pi(T)P = Vx \otimes Vy$.

Indeed: $P\pi(T)P = VV^*\pi(T)VV^* = VTV^* = V(x \otimes y)V^*$ since $T = V^*\pi(T)V$.

For any $w \in \mathcal{H}$, $V(x \otimes y)V^*w = V[(V^*w,x)y] = (w,Vx)Vy = (Vx \otimes Vy)(w)$. Thus $P\pi(T)P = V(x \otimes y)V^* = Vx \otimes Vy$.

By our hypotheses, $C^*(\mathfrak{J})$ is an irreducible C^*-algebra which contains a non-zero compact operator; it must therefore contain the algebra of all compact operators, $\mathfrak{BC}(\mathcal{H})$. [2, Corollary 4.1.10] $\mathfrak{BC}(\mathcal{H})$ is then an ideal in $C^*(\mathfrak{J})$, so by lemma 22 we can write $\pi = \pi_o \oplus \pi_1$ where π_o is a representation of $C^*(\mathfrak{J})$ which annihilates compact operators and π_1 is the unique extension to $C^*(\mathfrak{J})$ of a non-degenerate representation of $\mathfrak{B}\mathcal{C}(\mathcal{H})$. Note that π_1 must be a multiple of the identity representation. Let $\mathcal{K}_o$, $\mathcal{K}_1$ be the subspaces of $\mathcal{K}$ on which π_o and π_1 act, respectively. (So $\mathcal{K} = \mathcal{K}_o \oplus \mathcal{K}_1$.) Let Q_o, Q_1 be orthogonal projection on $\mathcal{K}_o$, $\mathcal{K}_1$ respectively.

<u>Observation 2</u>. $\pi_o = 0$.

First suppose that x and y are unit vectors such that $T = x \otimes y \in \mathcal{S}$. Then, using the facts that $\pi_o(T) = 0$ and π_1 is norm-decreasing, we have

$$\begin{aligned} 1 = \|Vy\| &= \|(Vx \otimes Vy)Vx\| = \|P\pi(T)PVx\| \\ &\leq \|\pi(T)Vx\| = \|\pi_o(T)Q_oVx + \pi_1(T)Q_1Vx\| \\ &= \|\pi_1(T)Q_1Vx\| \leq \|Q_1Vx\| \leq \|Vx\| = 1 . \end{aligned}$$

Thus $\|Q_1Vx\| = 1$ and so $Q_oVx = 0$ and $Vx \in \mathcal{K}_1$. It immediately follows that if x is any vector such that $x \otimes y \in \mathcal{S}$ for some non-zero vector y, then $Vx \in \mathcal{K}_1$. In particular, if x is an arbitrary vector in $\mathcal{H}$ then VEx and $V(I - E)x$ lie in $\mathcal{K}_1$. Hence $Vx = VEx + V(I - E)x \in \mathcal{K}_1$ for all $x \in \mathcal{H}$. Thus $V\mathcal{H} \subseteq \mathcal{K}_1$. Since $\mathcal{K}_1$ is invariant under $\pi(C^*(\mathcal{J}))$ and $V\mathcal{H}$ is cyclic under $\pi(C^*(\mathcal{J}))$ we have $\mathcal{K} = \mathcal{K}_1$ and, consequently, $\mathcal{K}_o = (0)$ and $\pi_o = 0$.

We now know that $\pi = \pi_1$ = a multiple of the identity representation, so we can write $\pi = \sum_{i \in I} \oplus \pi_i$ where I is some index set and each $\pi_i \cong \mathrm{Id}$ = identity representation of $C^*(\mathfrak{J})$. For each $i \in I$, let $\mathcal{H}_i$ be the subspace of $\mathcal{H}$ on which π_i acts, let Q_i be orthogonal projection in $\mathfrak{B}(\mathcal{H})$ on $\mathcal{H}_i$, and let $U_i\colon \mathcal{H} \longrightarrow \mathcal{H}_i$ be the isometric isomorphism which implements the equivalence $\pi_i \cong \mathrm{Id}$. Note that for each $x \in \mathcal{H}$, $Vx = \sum_{i \in I} Q_i Vx$ and that if $T = x \otimes y \in C^*(\mathfrak{J})$ then $\pi_i(T) = U_i x \otimes U_i y$.

<u>Observation 3</u>. Let x be any non-zero vector in $\mathcal{H}$ and assume that, for some non-zero vector y, $x \otimes y \in \mathcal{S}$. Then there exists a unique family of scalars $t_{i,x}$ such that $Vx = \sum_{i \in I} t_{i,x} U_i x$.

The uniqueness of the scalars follows from the fact that the $U_i x$ are pairwise orthogonal vectors. It is sufficient to prove the observation for unit vectors. Indeed, if $x \otimes y \in \mathcal{S}$ then so does $\frac{1}{\|x\|}(x \otimes y) = \frac{x}{\|x\|} \otimes y$. If we multiply the equation $V\frac{x}{\|x\|} = \sum_{i \in I} t_{i,\frac{x}{\|x\|}} U_i\left(\frac{x}{\|x\|}\right)$ by $\|x\|$ we obtain $Vx = \sum_{i \in I} t_{i,\frac{x}{\|x\|}} U_i x$. So we assume x and y are unit vectors such that $x \otimes y \in \mathcal{S}$ (y could be replaced by $y/\|y\|$ if necessary) and we prove the existence of the family of scalars $t_{i,x}$ with the required property. As in observation 2,

$$1 = \|Vy\| = \|(Vx \otimes Vy)Vx\| = \|P\pi(T)PVx\|$$

$$= \|P\pi(T)Vx\| \leq \|\pi(T)Vx\| \leq \|Vx\| = 1$$

from which it follows that $\|\pi(T)Vx\| = 1$. (Note that we can also deduce $P\pi(T)Vx = \pi(T)Vx$ for $T = x \otimes y \in \mathcal{S}$. We shall need this shortly.) Now, using

$\pi(T)Vx = \pi(T) \sum_{i\in I} Q_i Vx = \sum_{i\in I} \pi_i(T)Q_iVx$, we obtain the following equality:

$$1 = \|\pi(T)Vx\|^2 = \|\sum_{i\in I} \pi_i(T)Q_iVx\|^2$$

$$= \sum_{i\in I} \|\pi_i(T)Q_iVx\|^2 = \sum_{i\in I} \|(U_ix \otimes U_iy)Q_iVx\|^2$$

$$= \sum_{i\in I} \|(Q_iVx,U_ix)U_iy\|^2 = \sum_{i\in I} |(Q_iVx,U_ix)|^2$$

$$\leq \sum_{i\in I} \|Q_iVx\|^2 = \|\sum_{i\in I} Q_iVx\|^2 = \|Vx\|^2 = 1 .$$

So $\sum_{i\in I} |(Q_iVx,U_ix)|^2 = \sum_{i\in I} \|Q_iVx\|^2$. Since for each i, $|(Q_iVx,U_ix)| \leq \|Q_iVx\|$ we must have $|(Q_iVx,U_ix)| = \|Q_iVx\|$ and hence $Q_iVx = (Q_iVx,U_ix)U_ix$. Let $t_{i,x} = (Q_iVx,U_ix)$. Then $Vx = \sum_{i\in I} Q_iVx = \sum_{i\in I} t_{i,x}U_ix$.

If $x \otimes y \in \mathcal{S}$ and $x \neq 0$, $y \neq 0$ then there exist unique scalars $t_{i,y}$ such that $Vy = \sum_{i \in I} t_{i,y} U_i y$. Indeed, just observe that $y \otimes x = (x \otimes y)^* \in \mathcal{S}$ and apply observation 3.

<u>Observation 4</u>. Let $T = x \otimes y \in \mathcal{S}$ with x and y non-zero. Then for each $i \in I$, $t_{i,x} = t_{i,y}$.

As before we may assume $\|x\| = \|y\| = 1$. By the parenthetical remark in the middle of observation 3, $P\pi(T)Vx = \pi(T)Vx$.

$$\begin{aligned}\sum_{i \in I} t_{i,y} U_i y = Vy &= (Vx \otimes Vy)Vx = P\pi(T)PVx \\ &= \pi(T)Vx = \sum_{i \in I} \pi(T) t_{i,x} U_i x \\ &= \sum_{i \in I} t_{i,x} \pi_i(T) U_i x = \sum_{i \in I} t_{i,x} U_i y\end{aligned}$$

(Recall: $\pi_i(T) = U_i x \otimes U_i y$.) Since the $U_i y$ are a pairwise orthogonal family of vectors, $t_{i,x} = t_{i,y}$ all i.

Note that as a consequence of observation 4 and the hypotheses if x is a vector such that $Ex \neq 0$ and $(I - E)x \neq 0$ then $t_{i,Ex} = t_{i,(I - E)x}$ for all i.

<u>Observation 5</u>. If x is an arbitrary non-zero vector in $\mathcal{H}$ then there exists a unique family of scalars $t_{i,x}$ such that $Vx = \sum_{i \in I} t_{i,x} U_i x$.

If $Ex = 0$ or if $(I - E)x = 0$ then observation 3 may be applied to x. If $Ex \neq 0$ and $(I - E)x \neq 0$ then, letting $t_i = t_{i,Ex} = t_{i,(I-E)x}$, we have $VEx = \sum_{i \in I} t_i U_i Ex$ and $V(I - E)x = \sum_{i \in I} t_i U_i (I - E)x$. So

$$\begin{aligned} Vx &= VEx + V(I - E)x \\ &= \sum_{i \in I} t_i U_i Ex + \sum_{i \in I} t_i U_i (I - E)x \\ &= \sum_{i \in I} t_i U_i x \end{aligned}$$

and we may take $t_{i,x} = t_i$.

<u>Observation 6</u>. Let x, y be non-zero vectors in $\mathcal{H}$. Then $t_{i,x} = t_{i,y}$ for all $i \in I$.

Suppose first that $Ex \neq 0$. If $(I - E)y \neq 0$ also, then $Ex \otimes (I - E)y \in \mathcal{S}$ and $t_{i,x} = t_{i,Ex} = t_{i,(I-E)y} = t_{i,y}$. If $(I - E)y = 0$ then $Ey \neq 0$ and, for an arbitrary non-zero vector $w \in (I - E)(\mathcal{H})$, both $Ex \otimes w$ and $Ey \otimes w$ lie in $\mathcal{S}$. Hence $t_{i,x} = t_{i,Ex} = t_{i,w} = t_{i,Ey} = t_{i,y}$. Thus in either case $t_{i,x} = t_{i,y}$. On the other hand, if $Ex = 0$ then $(I - E)x \neq 0$ and essentially the same argument proves $t_{i,x} = t_{i,y}$.

Thus $t_{i,x}$ is independent of the vector x, so we write $t_i = t_{i,x}$. We then obtain the formula

$$Vx = \sum_{i \in I} t_i U_i x \qquad \text{all} \quad x \in \mathcal{H}.$$

With this formula we can easily finish the proof of the theorem. Let $T \in C^*(\mathfrak{T})$ be arbitrary.

$$\pi(T)Vx = \sum_{i \in I} \pi(T)(t_i U_i x) = \sum_{i \in I} t_i \pi_i(T) U_i x$$

$$= \sum_{i \in I} t_i U_i Tx = VTx \,. \qquad (\text{all} \quad x \in \mathcal{H}).$$

The next to the last equality follows from the fact that U_i implements the equivalence $\pi_i \cong \mathrm{Id}$.

Thus $V\mathcal{H}$ is left invariant by $\pi(C^*(\mathfrak{T}))$. Since it is also cyclic under $\pi(C^*(\mathfrak{T}))$, $V\mathcal{H} = \mathcal{K}$, which is what we needed to show.

Corollary 24. Let $\mathfrak{R}$ and $\mathfrak{T}$ be linear subspaces of operators on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ respectively. Assume that $\mathfrak{R}$ and $\mathfrak{T}$ each contain the identity operator and that each satisfies the hypothesis of Theorem 23. Let $\varphi: \mathfrak{R} \longrightarrow \mathfrak{T}$ be a completely isometric linear map of $\mathfrak{R}$ on $\mathfrak{T}$ such that $\varphi(I) = I$. Then φ is implemented by a unitary mapping of the underlying Hilbert spaces.

Proof. By Theorem 23 the identity representations of $C^*(\mathfrak{R})$ and $C^*(\mathfrak{T})$ are boundary representations for $\mathfrak{R}$ and

$\mathfrak{J}$, respectively. It follows therefore that $\mathfrak{R}$ and $\mathfrak{J}$ are admissible subspaces of $C^*(\mathfrak{R})$ and $C^*(\mathfrak{J})$ and that each has Silov boundary ideal $\{0\}$. Hence, by theorem 21, φ is implemented by a $*$-isomorphism, which we again denote by φ, of $C^*(\mathfrak{R})$ on $C^*(\mathfrak{J})$. Now since $C^*(\mathfrak{R})$ is irreducible and contains at least one non-zero compact operator, we have $\mathfrak{BC}(\mathfrak{H}) \subseteq C^*(\mathfrak{R})$. Similarly $\mathfrak{BC}(\mathfrak{K}) \subseteq C^*(\mathfrak{J})$. Further, we must have $\varphi(\mathfrak{BC}(\mathfrak{H})) = \mathfrak{BC}(\mathfrak{K})$ since the compact operators form a unique minimal closed ideal in any C*-algebra containing them. Thus φ restricted to $\mathfrak{BC}(\mathfrak{H})$ is an irreducible representation of the compact operators on $\mathfrak{H}$, hence φ is unitarily equivalent to the identity representation. That is, there is a unitary mapping U of $\mathfrak{K}$ onto $\mathfrak{H}$ such that $\varphi(S) = U^{-1}SU$ for all $S \in \mathfrak{BC}(\mathfrak{H})$. Since there is a unique non-degenerate representation of $C^*(\mathfrak{R})$ extending the restriction of φ to $\mathfrak{BC}(\mathfrak{H})$, we can conclude that $\varphi(S) = U^{-1}SU$ for all $S \in C^*(\mathfrak{R})$. Thus φ is unitarily implemented.

<u>Corollary 25</u>. Let $\mathfrak{R}$ (respectively $\mathfrak{J}$) be either a nest algebra or a large subalgebra of a nest algebra or a non-irreducible maximal triangular algebra. Then any completely isometric linear mapping of $\mathfrak{R}$ on $\mathfrak{J}$ which

preserves the identities is unitarily implemented.

Proof. It is trivial that nest algebras and large subalgebras of nest algebras satisfy the hypotheses of Corollary 24. That any non-irreducible maximal triangular algebra also satisfies these hypotheses follows from Lemma 2.

Remark. The word "completely" is necessary in the hypotheses of Corollary 25, even in finite dimensions. If $\mathcal{H}$ is a finite dimensional Hilbert space, by choosing an orthonormal basis in a fixed order we may identify $\mathfrak{B}(\mathcal{H})$ with M_n. The algebras $\mathcal{S}$ and $\mathcal{J}$ of all upper and all lower triangular matrices, respectively, are ordered bases in M_n and the map $A \longrightarrow A^t$ = transpose of A is a linear isometry of $\mathcal{S}$ on $\mathcal{J}$ which carries the identity to the identity. Since $A \longrightarrow A^t$ is anti-multiplicative it cannot be implemented by a unitary transformation. It follows that $A \longrightarrow A^t$ is not completely isometric. (To see that $A \longrightarrow A^t$ is an isometry, let $\bar{A} = (\bar{a}_{ij})$ for each matrix $A = (a_{ij})$. Since $A^t = \bar{A}^*$ and $A \longrightarrow A^*$ is an isometry, it suffices to show that $\|A\| = \|\bar{A}\|$ for all matrices A. If $x = (x_1,\ldots,x_n)$ is a unit vector then

$\bar{x} = (\bar{x}_1,\ldots,\bar{x}_n)$ is a unit vector also, and $\bar{A}\bar{x} = (\overline{Ax})$, which has the same norm as the vector Ax. Hence $\|A\| = \|\bar{A}\|$.)

It should be remarked, however, that in finite dimensions a "transpose" map is essentially the only possible isometry which is not completely isometric. More specifically, suppose $\mathcal{S} \subseteq M_n$ and $\mathcal{T} \subseteq M_n$ are each the algebra of all upper triangular matrices with respect to suitable bases, and suppose that φ is a linear isometry of $\mathcal{S}$ onto $\mathcal{T}$ such that $\varphi(I) = I$. Then, since $M_n = \mathcal{T} + \mathcal{T}^*$ it follows (see [1], Prop. 1.2.8) that φ has a unique extension to a positive linear map from M_n onto M_n. It is easy to see (by considering φ^{-1}) that the extension is actually an isometry. But then since $\varphi(I) = I$ and M_n is a factor, it follows from results in [6] that φ is either a *-isomorphism or a *-anti-isomorphism. A *-isomorphism is, of course, completely isometric and a *-anti-isomorphism can easily be written as the product of a *-isomorphism and a "transpose" map.

III. IDEALS IN ORDERED BASES

On a finite dimensional Hilbert space an ordered basis $\mathfrak{J}$ consists of the algebra of all upper triangular matrices (relative to a fixed basis taken in a fixed order). The radical $\mathfrak{R}$ of $\mathfrak{J}$ is then the algebra of strictly upper triangular matrices. $\mathfrak{R}$ is a (two-sided) ideal in $\mathfrak{J}$ with the property that $\mathfrak{R} \cap \mathfrak{D} = (0)$, where $\mathfrak{D}$, the diagonal of $\mathfrak{J}$, is the algebra of all diagonal matrices. $\mathfrak{R}$ is the unique ideal in $\mathfrak{J}$ maximal with respect to this property. Furthermore, every operator in $\mathfrak{J}$ can be uniquely decomposed as the sum of an operator in $\mathfrak{D}$ and an operator in $\mathfrak{R}$.

Such a decomposition is probably too much to hope for in the general infinite dimensional case. It is certain that the radical cannot play the same role in general as it does in finite dimensions.

John von-Neumann was the first to introduce a process for taking the "diagonal part" of linear operators. In [4] Kadison and Singer make the following definition (presented here in a restricted form).

<u>Definition 26</u>. Let $\mathfrak{D}$ be a maximal abelian self-adjoint subalgebra of $\mathfrak{B}(\mathcal{H})$. A <u>diagonal process relative to $\mathfrak{D}$</u> is a linear, order preserving mapping of $\mathfrak{B}(\mathcal{H})$ into $\mathfrak{D}$ which is the identity on $\mathfrak{D}$.

Kadison and Singer prove the existence of diagonal processes (for $\mathcal{H}$ separable) and prove that diagonal processes possess the following crucial property.

<u>Proposition 27</u>. Let φ be a diagonal process relative to $\mathfrak{D}$. Then for any $X \in \mathfrak{B}(\mathcal{H})$, $D \in \mathfrak{D}$, $\varphi(XD) = \varphi(X)D$ and $\varphi(DX) = D\varphi(X)$.

If $\mathfrak{T}$ is an ordered basis with diagonal $\mathfrak{D}$, by a "diagonal process on $\mathfrak{T}$" we simply mean the restriction to $\mathfrak{T}$ of a diagonal process on $\mathfrak{B}(\mathcal{H})$ relative to $\mathfrak{D}$. The following definitions of diagonal disjoint ideals and φ-diagonal zero ideals and Proposition 30 are taken from an unpublished fourth chapter to [5].

<u>Definition 28</u>. An ideal $\mathcal{J}$ in an ordered basis $\mathfrak{T}$ with diagonal $\mathfrak{D}$ is called a <u>diagonal disjoint</u> ideal if $\mathcal{J} \cap \mathfrak{D} = (0)$. $\mathcal{J}$ is a <u>maximal diagonal disjoint ideal</u> if it is not properly contained in another diagonal disjoint

ideal.

Definition 29. Let φ be a diagonal process on an ordered basis $\mathfrak{J}$. An ideal $\mathcal{J}$ in $\mathfrak{J}$ is called a φ-diagonal zero ideal if $\varphi(J) = 0$ for all $J \in \mathcal{J}$. $\mathcal{J}$ is a maximal φ-diagonal zero ideal if it is not properly contained in any other φ-diagonal zero ideal.

Proposition 30. Let $\mathfrak{J}$ be an ordered basis with diagonal $\mathcal{D}$. Each diagonal disjoint ideal is contained in a maximal diagonal disjoint ideal. If φ is a diagonal process on $\mathfrak{J}$ then there is a maximal φ-diagonal zero ideal which contains every other φ-diagonal zero ideal and which is also a diagonal disjoint ideal. If E and $I - F$ are hulls of $\mathfrak{J}$ then $E\mathfrak{J}F$ is an ideal in $\mathfrak{J}$. If, further, $EF = 0$ then $E\mathfrak{J}F$ is contained in each maximal ideal of the two types defined above.

Proof. The union of an ascending chain of diagonal disjoint ideals is again diagonal disjoint; so Zorn's lemma implies the existence of a maximal diagonal disjoint ideal. The existence of a maximal φ-diagonal zero ideal is proven in the same way. Since φ is additive, the sum of two φ-diagonal zero ideals is again a φ-diagonal zero ideal;

hence a maximal φ-diagonal zero ideal contains any other φ-diagonal zero ideal (and is therefore unique). If $\mathcal{J}$ is any φ-diagonal zero ideal and $D \in \mathcal{J} \cap \mathcal{D}$ then $0 = \varphi(D) = D$. Thus $\mathcal{J} \cap \mathcal{D} = (0)$ and $\mathcal{J}$ is diagonal disjoint.

Clearly $E\mathfrak{J}F$ is a linear subspace of $\mathfrak{J}$. If B is any operator in $\mathfrak{J}$ then $BE = EBE$ and $FB = FBF$ since E and $I - F$ are hulls. Then for any element $ETF \in E\mathfrak{J}F$, we have

$$BETF = EBETF \in E\mathfrak{J}F$$

and

$$ETFB = ETFBF \in E\mathfrak{J}F$$

Thus $E\mathfrak{J}F$ is an ideal in $\mathfrak{J}$.

Now suppose that $EF = 0$. Then for any $T \in \mathfrak{J}$, $\varphi(ETF) = E\varphi(T)F = EF\varphi(T) = 0$. Thus $E\mathfrak{J}F$ is a φ-diagonal zero ideal and is, of course, contained in the maximal one.

Finally, we show that if $\mathcal{J}$ is a maximal diagonal disjoint ideal and if $EF = 0$ then $E\mathfrak{J}F \subseteq \mathcal{J}$. From the maximality of $\mathcal{J}$ it will suffice to prove that $\mathcal{J} + E\mathfrak{J}F$ is diagonal disjoint. Suppose that $B \in \mathcal{J}$, $T \in \mathfrak{J}$ and $D = B + ETF \in \mathcal{D}$. Then

$D(I - E) = (I - E)D = (I - E)B + (I - E)ETF = (I - E)B \in \mathcal{J}$ and $DE = BE + ETFE = BE \in \mathcal{J}$. Hence $D = DE + D(I - E) \in \mathcal{J}$ and $D \in \mathfrak{D}$, so $D = 0$. Thus $(\mathcal{J} + E\mathfrak{J}F) \cap \mathfrak{D} = (0)$.

For the following results we shall use the fact that a positive linear map on a C*-algebra which preserves the identity has norm equal to one. (See [11], page 415)

<u>Lemma 31</u>. Let $\mathfrak{J}$ be an ordered basis with diagonal $\mathfrak{D}$. Let φ be a diagonal process on $\mathfrak{J}$ and let $\mathfrak{K}$ be the maximal φ-diagonal zero ideal. Then $\mathfrak{K}$ is closed in the uniform topology.

<u>Proof</u>. Since $\mathfrak{J}$ is closed in the uniform topology, $\overline{\mathfrak{K}} \subseteq \mathfrak{J}$. If $T \in \overline{\mathfrak{K}}$ then T can be uniformly approximated by operators which are annihilated by φ. But φ is continuous, so $\varphi(T) = 0$ also. Thus $\overline{\mathfrak{K}}$ is a φ-diagonal zero ideal containing $\mathfrak{K}$, hence $\overline{\mathfrak{K}} = \mathfrak{K}$.

<u>Proposition 32</u>. Let $\mathfrak{J}$, $\mathfrak{D}$, φ and $\mathfrak{K}$ be as in Lemma 31 and let $\mathfrak{R}$ be the radical of $\mathfrak{J}$. Then $\mathfrak{R} \subseteq \mathfrak{K}$.

<u>Proof</u>. Since $\mathfrak{R}$ is an ideal in $\mathfrak{J}$ we need only show $\varphi(T) = 0$ for $T \in \mathfrak{R}$. Let $\varepsilon > 0$ be arbitrary. By Ringrose's criterion (Theorem 9) there exists a nest of

hulls $0 = E_o < E_1 < \ldots < E_n = I$ in $\mathfrak{D}$ such that $\|(E_i - E_{i-1})T(E_i - E_{i-1})\| < \varepsilon$ for each $i = 1,\ldots,n$. Let $S = \sum_{i=1}^{n} (E_i - E_{i-1})T(E_i - E_{i-1})$. Then

$$\|S\| = \max_{i=1,\ldots,n} \|(E_i - E_{i-1})T(E_i - E_{i-1})\| < \varepsilon$$

Now
$$\varphi(S) = \sum_{i=1}^{n} (E_i - E_{i-1})\varphi(T)(E_i - E_{i-1})$$

$$= \sum_{i=1}^{n} (E_i - E_{i-1})\varphi(T)$$

$$= \varphi(T)$$

hence $\|\varphi(T)\| = \|\varphi(S)\| \leq \|S\| < \varepsilon$. As ε was arbitrary, $\|\varphi(T)\| = 0$.

<u>Lemma 38</u>. Let $\mathfrak{T}$ be an ordered basis with diagonal $\mathfrak{D}$ and let $\mathcal{J}$ be a maximal diagonal disjoint ideal. Then $\mathcal{J}$ is closed in the uniform topology.

<u>Proof</u>. The closure $\overline{\mathcal{J}}$ of $\mathcal{J}$ is an ideal in $\mathfrak{T}$. Suppose $\overline{\mathcal{J}} \cap \mathfrak{D} \neq (0)$. Then $\overline{\mathcal{J}} \cap \mathfrak{D}$ is a non-zero ideal in the abelian von-Neumann algebra $\mathfrak{D}$ and must therefore contain a non-zero projection P. ($\mathfrak{D} \cong C(X)$ where X is some stonean space. Let $f \neq 0$ be an element in $\overline{\mathcal{J}} \cap \mathfrak{D}$. Then $|f| \geq \varepsilon$ on some non-empty open and closed subset V

of X. If χ_V is the characteristic function of V then $\frac{1}{f}\chi_V$ is continuous and $(\frac{1}{f}\chi_V)f = \chi_V$ is a projection in $\bar{\mathcal{J}} \cap \mathfrak{D}$.) Let q be a real number such that $0 < q < 1$. Then there exists an element $J \in \mathcal{J}$ such that $\|P - J\| < q$. PJP also lies in $\mathcal{J}$ and $\|P - PJP\| \leq \|P\|\|P - J\|\|P\| = \|P - J\| < q$. The series $P + (P - PJP) + (P - PJP)^2 + \ldots$ is uniformly convergent; let H be its sum. Clearly, $H \in \mathfrak{J}$. Since

$$[P - (P - PJP)][P + (P - PJP) + \ldots + (P - PJP)^N]$$
$$= P - (P - PJP)^{N+1}$$

and $\lim_{N\to\infty} (P - (P - PJP)^{N+1}) = P$ (uniform limit) we conclude that $(PJP)H = P$, and in particular, $P \in \mathcal{J}$. Since $\mathcal{J} \cap \mathfrak{D} = (0)$ we must have $P = 0$, a contradiction. Thus $\bar{\mathcal{J}} \cap \mathfrak{D} = (0)$ and $\bar{\mathcal{J}}$ is a diagonal disjoint ideal containing the maximal diagonal disjoint ideal $\mathcal{J}$. We then have $\bar{\mathcal{J}} = \mathcal{J}$ and $\mathcal{J}$ is uniformly closed.

<u>Proposition 34</u>. Let $\mathfrak{J}$, $\mathfrak{D}$ and $\mathcal{J}$ be as in Lemma 33 and let $\mathfrak{R}$ be the radical of $\mathfrak{J}$. Then $\mathfrak{R} \subseteq \mathcal{J}$.

<u>Proof</u>. Since $\mathcal{J}$ and $\mathfrak{R}$ are ideals $\mathcal{J} + \mathfrak{R}$ is an ideal containing $\mathcal{J}$ and it suffices to prove that $\mathcal{J} + \mathfrak{R}$

is diagonal disjoint. Let $J \in \mathcal{J}$ and $R \in \mathcal{R}$ be such that $D = J + R \in \mathcal{D}$. Let $\varepsilon > 0$ be arbitrary. By Ringrose's criterion there is a nest of hulls

$0 = E_o < E_1 < \dots < E_n = I$ such that

$\|(E_i - E_{i-1})R(E_i - E_{i-1})\| < \varepsilon$ for $i = 1,2,\dots,n$. The family of projections $\{E_i - E_{i-1}\}_{i=1,\dots,n}$ is mutually orthogonal, so

$$\left\|\sum_{i=1}^{n} (E_i - E_{i-1})R(E_i - E_{i-1})\right\|$$

$$= \sup_{i=1,\dots,n} \|(E_i - E_{i-1})R(E_i - E_{i-1})\| < \varepsilon .$$

But

$$\sum_{i=1}^{n} (E_i - E_{i-1})R(E_i - E_{i-1})$$

$$= \sum_{i=1}^{n}(E_i - E_{i-1})D(E_i - E_{i-1}) - \sum_{i=1}^{n}(E_i - E_{i-1})J(E_i - E_{i-1})$$

$$= D - \sum_{i=1}^{n} (E_i - E_{i-1})J(E_i - E_{i-1})$$

and $\sum_{i=1}^{n} (E_i - E_{i-1})J(E_i - E_{i-1}) \in \mathcal{J}$. Thus D may be uniformly approximated by elements of $\mathcal{J}$. By the preceding lemma, $\mathcal{J}$ is closed, so $D \in \mathcal{J}$. Since $\mathcal{J}$ is diagonal disjoint, $D = 0$. Thus $\mathcal{J} + \mathcal{R}$ is diagonal disjoint.

We now provide an example in which the radical is a proper subset of the intersection of all maximal ideals of either type. Let $\mathfrak{T}$ be the standard ordered basis based on Lebesgue measure on the unit interval and use the notation of page 3. Define a bounded linear operator W mapping $L^2(0,1)$ into $L^2(0,1)$ by

$$Wf(t) = \begin{cases} f(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ 0 & \text{if } t > \frac{1}{2} \end{cases}$$

for each $f \in L^2(0,1)$. Observe that $WE_t = E_{\frac{1}{2}t}WE_t$ for all $t \in [0,1]$. Indeed, it follows trivially from the definition of W that if $f = 0$ almost everywhere on $[t,1]$ then $Wf = 0$ almost everywhere on $[t/2,1]$. A consequence is that $W \in \mathfrak{T}$. This property is retained by each element in the ideal $\mathfrak{J}$ generated by W. For, if $S \in \mathfrak{J}$ then S is of the form $S = \sum_{i=1}^{n} R_i W T_i$ where $R_i, T_i \in \mathfrak{T}$. Then

$$
\begin{aligned}
SE_t &= \sum_{i=1}^{n} R_i W T_i E_t = \sum_{i=1}^{n} R_i W E_t T_i E_t \\
&= \sum_{i=1}^{n} R_i E_{\frac{1}{2}t} W E_t T_i E_t \\
&= \sum_{i=1}^{n} E_{\frac{1}{2}t} R_i E_{\frac{1}{2}t} W T_i E_t \\
&= \sum_{i=1}^{n} E_{\frac{1}{2}t} R_i W T_i E_t \\
&= E_{\frac{1}{2}t} S E_t
\end{aligned}
$$

Using Ringrose's criterion it is easy to see that W does not lie in the radical, $\mathfrak{R}$, of $\mathfrak{T}$. In fact, it will be sufficient to show that for any $s > 0$, $\|E_s W E_s\| \geq \sqrt{\frac{1}{2}}$. If $0 < t \leq s$ then $W(\chi_{[0,t]}) = \chi_{[0,\frac{1}{2}t]}$ and $\|\chi_{[0,\frac{1}{2}t]}\| = \sqrt{\frac{1}{2}}\, t^{\frac{1}{2}} = \sqrt{\frac{1}{2}}\, \|\chi_{[0,t]}\|$.

We claim that W lies in each maximal diagonal disjoint ideal, or, equivalently, that $\mathfrak{J}$ is contained in each maximal diagonal disjoint ideal. Let $\mathcal{J}$ be a maximal diagonal disjoint ideal: we must show that $(\mathcal{J} + \mathfrak{J}) \cap \mathfrak{D} = (0)$. Let $J \in \mathcal{J}$, $S \in \mathfrak{J}$ be such that $D = J + S \in \mathfrak{D}$. We will prove, by induction, that $D = DE_{2^{-n}}$ for each $n = 0,1,2,\ldots,$ from which it follows that $D = \text{strong}\lim DE_{2^{-n}} = 0$. For $n = 0$ the statement is trivial. Assume $D = DE_{2^{-k}}$. Then

$DE_{2^{-k}} = JE_{2^{-k}} + SE_{2^{-k}} = JE_{2^{-k}} + E_{2^{-k-1}}SE_{2^{-k}}$, whence $(E_{2^{-k}} - E_{2^{-k-1}})DE_{2^{-k}} = (E_{2^{-k}} - E_{2^{-k-1}})JE_{2^{-k}}$. Since the left side of this equation lies in $\mathfrak{D}$ and the right side lies in $\mathfrak{J}$, both sides are 0 and $D = DE_{2^{-k}} = DE_{2^{-k-1}}$.

Finally, if φ is a diagonal process on $\mathfrak{J}$ then W lies in the maximal φ-diagonal zero ideal. It suffices to show that φ vanishes on $\mathfrak{J}$. If $S \in \mathfrak{J}$ then

$$\varphi(S) = \varphi(E_{1/2}S) = E_{1/2}\varphi(S) = \varphi(S)E_{1/2}$$
$$= \varphi(SE_{1/2}) = \varphi(E_{1/4}SE_{1/2}) = E_{1/4}\varphi(S) = \ldots$$

and, by induction, $\varphi(S) = E_{2^{-n}}\varphi(S)$ for all positive integers n. Since $E_{2^{-n}}\varphi(S) \longrightarrow 0$ strongly as $n \to \infty$ we have $\varphi(S) = 0$.

A consequence of our final proposition will be that an operator unitarily equivalent to the Volterra integration operator lies in the radical of the standard ordered basis based on Lebesgue measure.

<u>Proposition 35</u>. Let $\mathfrak{J}$ be the standard ordered basis based on Lebesgue measure. Let $I = [0,1]$ and let $k \in L^2(I \times I)$. Assume $k(x,y) = 0$ whenever $y < x$.

Define an operator K by $Kf(x) = \int_x^1 k(x,y)f(y)\,dy$. Then K lies in the radical of $\mathfrak{I}$.

Proof. It is well known that K is a bounded linear operator on $L^2(0,1)$. (See, for example, [3, p. 135].) If f vanishes almost everywhere on $[t,1]$ then so does Kf, so $K \in \mathfrak{I}$. We show that K satisfies Ringrose's criterion. First, observe that for any $0 \leq t < s \leq 1$, the operator $(E_s - E_t)K(E_s - E_t)$ arises from the kernel function $Ch_{(t,s)}k$, where $Ch_{(t,s)}$ is the characteristic function of the set

$[t,s] \times [t,s] = \{(x,y) |\ t \leq x \leq s,\ t \leq y \leq s\}$. Indeed: if $f \in L^2(0,1)$ (and $\chi_{[t,s]}$ is the characteristic function of the interval $[t,s]$) then

$$
\begin{aligned}
(E_s - E_t)K(E_s - E_t)f(x) & \\
&= \chi_{[t,s]}(x)[K(E_s - E_t)f(x)] \\
&= \chi_{[t,s]}(x) \int_x^1 k(x,y)(E_s - E_t)f(y)\,dy \\
&= \chi_{[t,s]}(x) \int_x^1 k(x,y]\chi_{[t,s]}(y)f(y)\,dy \\
&= \int_x^1 \chi_{[t,s]}(x)\chi_{[t,s]}(y)k(x,y)f(y)\,dy \\
&= \int_x^1 [Ch_{(t,s)}(x,y)k(x,y)]f(y)\,dy\ .
\end{aligned}
$$

For each integer $n > 0$ let C_n be the characteristic function of $\{(x,y) \mid 0 \leq x, y \leq 1 \text{ and } x \leq y \leq x + 1/n\}$. Now, $|C_n k|^2 \to 0$ pointwise (almost everywhere) as $n \to \infty$ and $|C_n k|^2 \leq |k|^2$ for all n, so the Lebesgue dominated convergence theorem implies $\|C_n k\|_2 \to 0$ as $n \to \infty$ (in $L^2(I \times I)$). If $0 \leq t < s \leq 1$ and $s - t \leq 1/n$ then $|Ch_{(t,s)}k| \leq |C_n k|$ and hence $\|(E_s - E_t)K(E_s - E_t)\| = \|Ch_{(t,s)}k\|_2 \leq \|C_n k\|_2$. Given $\varepsilon > 0$, choose n sufficiently large that $\|C_n k\|_2 < \varepsilon$. Then the nest E_o $(= 0)$, $E_{1/n}$, $E_{2/n}$, $\ldots$, E_1 $(= I)$ satisfies the condition of Theorem 9 for K and ε:

$$\|(E_{\frac{h}{n}} - E_{\frac{h-1}{n}})K(E_{\frac{h}{n}} - E_{\frac{h-1}{n}})\| \leq \|C_n k\|_2 < \varepsilon \quad \text{for } h = 1,2,\ldots,n.$$

Thus $K \in \mathfrak{R}$.

<u>Remark</u>. If we let $k(x,y) = \begin{cases} 0 & \text{if } y < x \\ 1 & \text{if } y \geq x \end{cases}$

then $Kf(x) = \int_x^1 f(y)\,dy$ is the adjoint of the Volterra integration operator.

BIBLIOGRAPHY

1. Arveson, William B., Subalgebras of C*-algebras. Acta Mathematica, 123 (1969), 141-224.

2. Dixmier, Jacques, Les C*-algèbres et leur représentations. Gauthier-Villars, Paris. 2nd edition. 1969.

3. Halmos, Paul R., A Hilbert Space Problem Book. Van Nostrand, Princeton, N.J. 1967.

4. Kadison, R. V. and Singer, I. M., Extensions of Pure States. American Journal of Mathematics, 81 (1959), 383-400.

5. Kadison, R. V. and Singer, I. M., Triangular Operator Algebras. American Journal of Mathematics, 82 (1960), 227-259.

6. Kadison, R. V., Isometries of Operator Algebras, Annals of Mathematics, 54 (1951), 325-338.

7. von-Neumann, John, On Rings of Operators, III. Annals of Mathematics, 41 (1940), 94-161.

8. Ringrose, J. R., Algebraic Isomorphisms Between Ordered Bases. American Journal of Mathematics, 83 (1961), 463-478.

9. Ringrose, J. R., On Some Algebras of Operators. Proceedings of the London Mathematical Society (3) 15 (1965), 61-83.

10. Ringrose, J. R., On Some Algebras of Operators II. Proceedings of the London Mathematical Society (3) 16 (1966), 385-402.

11. Russo, B. and Dye, H. A., A Note on Unitary Operators in C*-algebras, Duke Mathematical Journal, 33 (1966), 413-416.

12. Stinespring, W. F., Positive Functions on C*-algebras. Proceedings of the American Mathematical Society, 6 (1955), 211-216.